Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 7, 2002*
Revised: *December 20, 2002*
Accepted: *January 9, 2003*

Renormalon disappearance in Borel sum of the $1/N$ expansion of the Gross-Neveu model mass gap

Jean-Loic Kneur and Damien Reynaud

Physique Mathématique et Théorique, UMR-5825-CNRS
Université Montpellier II, F–34095 Montpellier Cedex 5, France
E-mail: kneur@lpm.univ-montp2.fr, reynaud@lpm.univ-montp2.fr

Abstract: The exact mass gap of the $O(N)$ Gross-Neveu model is known, for arbitrary N, from non-perturbative methods. However, a "naive" perturbative expansion of the pole mass exhibits an infinite set of infrared renormalons at order $1/N$, formally similar to the QCD heavy quark pole mass renormalons, potentially leading to large $\mathcal{O}(\Lambda)$ perturbative ambiguities. We examine the precise vanishing mechanism of such infrared renormalons, which avoids this (only apparent) contradiction, and operates without need of (Borel) summation contour prescription, usually preventing unambiguous separation of perturbative contributions. As a consequence we stress the direct Borel summability of the $(1/N)$ perturbative expansion of the mass gap. We briefly speculate on a possible similar behaviour of analogous non-perturbative QCD quantities.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Lower Dimensions, Nonperturbative Effects, 1/N Expansion.

JHEP01(2003)014

Contents

JHEP01(2003)014

1. Introduction

The $(1+1)$ dimensional $O(N)$ Gross-Neveu (GN) model [1, 2] often serves as a simpler toy model for more complicated theories like QCD, sharing with it the properties of asymptotic freedom and dynamical mass generation, while being an integrable model with many exact results available. The exact mass gap (associated with the breaking of the discrete chiral symmetry), for arbitrary N, has been evaluated from non-perturbative (NP) methods, more precisely from exact S matrix results [3] associated with Thermodynamic Bethe Ansatz (TBA) methods [4]. Yet from a more general viewpoint, it can be an interesting issue (since still not fully clarified, in our opinion) to study the precise matching between those two-dimensional NP exact results on one side and the standard perturbative behaviour on the other side. This may give some more insight on the short/long distance physics interplay for more involved theories like four-dimensional QCD. In particular one apparent puzzle arises, once realizing, as we examine here, that the (naive) perturbative expansion of the pole mass of the (massive) GN model suffers at next-to-leading $1/N$ order from potential ambiguities, due to the presence of severe infrared renormalons, which are indeed formally completely similar to the quark pole mass renormalons [5]. While the TBA GN mass gap [4], and a fortiori its next-to-leading $1/N$ expansion [6], are unambiguously determined. Actually, independently of TBA results, it *is* expected on general grounds that any truly NP calculation should be free of such ambiguities [7], i.e. that infrared renormalons are perturbative artifacts. But up to now only a few NP results have been explored from this perspective, even for integrable models, thus we find instructive to examine in some details how exactly the NP contributions, here fully controllable at least at $1/N$ order, organize themselves so that the necessary vanishing of such ambiguities (indeed an infinite series of ambiguities) occurs. Note that, what is in fact generally expected (and illustrated in a few explicit calculations in the (1+1) dimensional sigma model [8, 9]) is that the complete NP result is unambiguous, but the separation of its perturbative from its NP "operator product expansion" (OPE) contribution, is not. More precisely the two contributions are intrinsically related through the need of a definite prescription in choosing the integration path in the Borel plane, if using e.g. Borel resummation methods, necessary to avoid the renormalons in both parts and to resum the otherwise ill-defined factorially divergent perturbative series. Now in contrast, as we will show, a more interesting aspect of our results is that no such prescription is needed for the NP infrared renormalon disappearance in the GN mass gap. More precisely, the mechanism is such that the perturbative (re)expansion (to be specified) of the pole mass at order $1/N$ defines a *directly* Borel summable new perturbative series, which can be thus unambiguously separated from the NP part.

The paper is organized as follows: in section 2, briefly recalling some known basic results of the GN calculations at next-to-leading $1/N$ order, we shall define more precisely our "naive" perturbative expansion of the $1/N$ order GN pole mass and exhibit its infrared renormalon divergences. In order to examine those issues, we will consider first the GN model with a non-zero mass term, and single out a contribution which corresponds to a purely perturbative information. (The ordinary perturbative expansion of the true mass gap being of course zero to all orders, since the mass gap is non-perturbative and of order $\Lambda \sim e^{-2\pi/(N g)}$ where g is the coupling). One can consistently recover the true mass gap in the massless lagrangian limit. In section 3, we rederive the known [10, 4] exact calculation of the GN mass gap at $1/N$ order, but in the light of the results of section 2. Namely, we stress the differences between this exact $1/N$ calculation, which automatically takes into account the full NP contributions, and the naive pole mass expansion of section 2 based on a purely perturbative information. We then perform in section 4 a detailed calculation, using a convenient (Mellin transform) method, to exhibit the precise interplay mechanism between purely perturbative and NP contributions within the $1/N$ expansion, and leading to the expected absence of renormalons in the complete NP mass gap. We give in section 5 conclusion and some more speculative comments on a possible generalization of those results to QCD. Finally, two short appendices are devoted to additional technical details.

2. Perturbative renormalons in the $1/N$ naive pole mass

We start by briefly recalling the standard construction [10, 4] of the mass gap of the $O(2N)$[1] GN model at order $1/N$, with a slightly different approach. Here we consider in fact the model with the addition of a lagrangian mass term, in order to define the pole mass in a somewhat more universal manner, making in particular the link with analogous quantities in other models (QCD typically) more transparent. As usual the true mass gap is to be considered in the chiral, massless lagrangian limit. The lagrangian thus reads

$$\mathcal{L}_{GN} = \overline{\Psi} i \partial\!\!\!/ \Psi - m \overline{\Psi}\Psi + \frac{g}{2}\left(\overline{\Psi}\Psi\right)^2 \tag{2.1}$$

[1] From now on, all expressions correspond to the $O(2N)$ model, for easiest comparison with the exact [4] and $1/N$ [6] mass gap results.

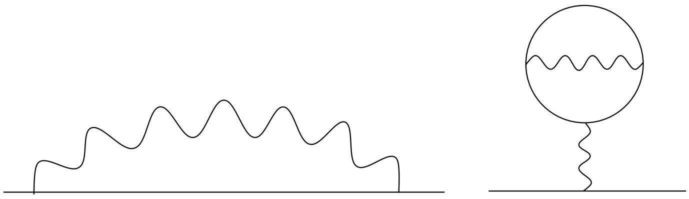

Figure 1: The GN mass graphs at order $1/N$.

and the graphs contributing to the two-point function at next-to-leading $1/N$ order are shown in figure 1. At this $1/N$ order, the renormalization procedure is relatively simple, since only the mass term is renormalized, which can be most simply performed by subtraction of the most divergent terms in a taylor expansion of the integrands [10]. The expression corresponding to figure 1 reads, after angular integration:

$$M_P = M\left[1+\frac{r_1}{N}+O\left(\frac{1}{N^2}\right)\right] \tag{2.2}$$

with $M = \Lambda \equiv \mu e^{-2\pi/(Ng(\mu))}$ the mass gap at leading $1/N$ order (μ is the arbitrary renormalization scale), and

$$\begin{aligned} r_1 = \frac{gN}{4\pi\, M}\Bigg[&\int_0^\infty dq^2 \left(\frac{\not p}{2p^2}\left[1-\frac{q^2+p^2+M_P^2}{A}\right]-\frac{M_P}{A}\right) G\left[q^2\right] - \\ &- 4G[0]M_P\int_0^\infty \frac{dq^2}{q^2\,\zeta}\ln\left[\frac{\zeta+1}{\zeta-1}\right] G\left[q^2\right]\Bigg] \end{aligned} \tag{2.3}$$

with $A \equiv \sqrt{(q^2+p^2+M_P^2)^2-4p^2\,M_P^2}$ (q is euclidean but p minkowskian), $\zeta \equiv (1+4M_P^2/q^2)^{1/2}$, and the dressed propagator (wavy line) is

$$G[q^2] = \left[1+\frac{gN}{2\pi}\left[\ln\frac{M_P^2}{\mu^2}+\zeta\ln\left[\frac{\zeta+1}{\zeta-1}\right]\right]\right]^{-1} = \left[\frac{gN}{2\pi}\zeta\ln\left[\frac{\zeta+1}{\zeta-1}\right]\right]^{-1}+\mathcal{O}\left(\frac{1}{N}\right). \tag{2.4}$$

To obtain the pole mass at the next-to-leading $1/N$ order, and in the massless limit, it is sufficient to replace $\not p = M_P = M[1+\mathcal{O}(1/N)]$ in eq. (2.3), (2.4) up to $1/N^2$ terms. This reads

$$r_1 = \frac{1}{4}\int_0^{\mu^2}\frac{dq^2}{M^2}(1-\zeta)\left[\zeta\ln\left[\frac{\zeta+1}{\zeta-1}\right]\right]^{-1} - \frac{1}{2}\int_0^{\mu^2}\frac{dq^2}{q^2\zeta^2}+\mathcal{O}\left(\frac{1}{N}\right) \quad - \text{subtraction} \tag{2.5}$$

where a "factorization" scale μ regularizing the integrals was introduced. The renormalization by subtraction will remove the divergent part of integrands, also regularized in term of μ. This sharp momentum cut-off procedure is rather similar to the one in ref. [6], giving results perfectly consistent with these authors, as we will see in next section, and is convenient for our purpose here, where we are primarily interested in the asymptotic behaviour of (2.5) (thus for $\mu \gg \Lambda$, but kept finite).

For simplicity we let aside in this section the tadpole second graph in figure 1, as its exact expression in (2.5) does not give any factorially divergent perturbative coefficients when expanded.[2] (It gives however a finite contribution to the mass gap, as we shall see in next section when a precise evaluation of the mass gap and its asymptotic behaviour will be considered.) We shall next examine in some detail how a naive perturbative expansion of (2.5) leads to infrared renormalons and related $\mathcal{O}(\Lambda/\mu)$ ambiguities. Before to proceed to see that, it is perhaps important to better explain what we exactly mean by "naive" perturbative expansion of (2.5). Clearly, the expression (2.5) is exact and complete at $1/N$ order, which means implicitly that it contains a priori *both* perturbative and NP contributions. One may argue that, in fact, the ordinary perturbative expansion of the true mass gap is zero to all orders, the mass gap being a NP quantity of order Λ. However, this is true strictly speaking only when considering the massless GN model, i.e. for $m \to 0$ in eq. (2.1). Alternatively, one may consider the calculation at this $1/N$ order of eq. (2.5) as valid for the massive GN lagrangian case where say, $m \sim M_P = \mathcal{O}(\Lambda)$ (for completeness, the case where $m \gg \Lambda$ is also briefly discussed in appendix A). Remark indeed that in (2.5), the $1-\zeta$ factor, which corresponds to the skeletal "one-loop" integrand of the first graph in figure 1, is essentially obtained from the pole mass condition $\not{p} = M_P$, irrespectively of whether M_P is $\mathcal{O}(\Lambda)$ or not. The remaining ζ–dependence is the dressed scalar propagator, as defined in eq. (2.4), where the leading $1/N$ order mass gap definition is only used in the right-hand side of eq. (2.4). Thus the perturbative form of (2.6) is quite generic, formally also applying to e.g. the QCD pole quark mass (but with obviously a different q^2–dependence replacing the $1-\zeta$ term). Now, since in the present case one can control everything at the $1/N$ order explicitly, our purpose is to consider the kind of results obtained when the NP knowledge is partially omitted. More precisely, let us assume that within the next-to-leading $1/N$ graph structure in figure 1, only the purely perturbative part of the dressed scalar propagator is known. Our motivation is that this is analogous to the situation in a more complicated theory like QCD typically. Starting thus from the "purely perturbative" information means replacing in (2.5) the dressed scalar propagator by the effective coupling:[3]

$$r_1 = \frac{1}{4}\int_0^{\mu^2} \frac{dq^2}{M^2}(1-\zeta)\,\left[g\left(q^2\right) - 2g\,\frac{M^2}{q^2} + \mathcal{O}\left(\frac{M^2}{q^2}\right)^2\right] \tag{2.6}$$

where in the bracket of eq. (2.6) the $\mathcal{O}(M^2/q^2)$ "NP" corrections to the effective coupling are of course exactly known in the GN case and can be obtained explicitly here from a systematic expansion of the right-hand side of eq. (2.4) for large q^2. Again, we stress that omitting this NP part is evidently not fully consistent, but the purpose of this (somewhat artificial) separation of the different contributions is to examine precisely how the renormalons (would) appear, in a way very similar to the QCD pole mass, and more importantly from which detailed mechanism do they finally disappear in the GN case, once the complete NP information is taken into account.

[2]Due to the second equality in (2.4), valid at this $1/N$ order, which makes the tadpole graph simplifying to purely polynomial integrals. Beyond the $1/N$ order, one could still in principle choose the arbitrary scale μ such that (2.4) holds.

[3]In the sequel we rescale $b_0\, g \equiv N/(2\pi)\, g \to g$, to define the $1/N$ expansion properly, and absorbing as well the 2π factor just for convenience.

The simplest standard procedure to exhibit the IR renormalons is by expanding the $1-\zeta$ term for small q^2:

$$1-\zeta = 1-\frac{2M}{q}\left(1+\frac{q^2}{4M^2}\right)^{1/2} \simeq 1-\frac{2M}{q}+\mathcal{O}\left(\frac{q}{2M}\right) \tag{2.7}$$

and expanding the effective coupling $g(q^2)$ at one loop order of renormalization group (RG) in powers of $b_0 g(\mu)\ln[q^2/\mu^2]$. From (2.7) it is seen that the leading singularity comes from the q^{-1} term which, combined with the $(g(\mu)\ln[q^2/\mu^2])^n$ terms of the effective coupling expansion, produces factorially divergent perturbative coefficients:

$$r_1^{\text{leading}} \sim -\frac{\mu}{M}\sum_{p=0}^{\infty} 2^p p!\, g^{p+1}(\mu)\,. \tag{2.8}$$

The non sign-alternation of those factorial coefficients implies [7] that the corresponding series is not Borel summable: the Borel integral corresponding to (2.8) reads

$$BI\left[r_1^{\text{leading}}\right] \sim -\frac{\mu}{M}\int_0^\infty dt e^{-t/g}(1-2t)^{-1} \tag{2.9}$$

so that the pole at $t_0 = 1/2$ on the integration path gives the leading ambiguity for the pole mass

$$\delta M_{\text{leading}} \sim \lim_{\epsilon\to 0}\left[\int^{\infty+i\epsilon} - \int^{\infty-i\epsilon}\right]\left(dt\, e^{-t/g}(1-2t)^{-1}\right) = \pm i\pi\mu e^{-t_0/g} \propto \Lambda\,. \tag{2.10}$$

Note that this leading renormalon ambiguity of $\mathcal{O}(\Lambda)$ is completely similar to the one derived for the quark pole mass in QCD [5]. The fact that the perturbative contribution eq. (2.9) is proportional to the factorization scale μ may appear at first puzzling (since it is supposed to be a contribution to the mass gap $\sim\Lambda\ll\mu$), it is in fact also a known feature of the non Borel summability of the corresponding series [7] with this explicit cutoff regularization: namely, it indicates us that an additional NP contribution, also proportionnal to the factorization scale, is necessarily needed (at least in this regularization scheme) in order to recover the right mass gap dependence $\sim\Lambda$.

Although our above derivation (starting from purely perturbative information) is very standard, the fact that the GN model perturbative *pole* mass at $1/N$ order also has the specific structure (2.8)–(2.10) of *infrared* renormalons, was perhaps not clearly appreciated before, to our knowledge.[4] Similarly, we can easily check from (2.3) that the leading asymptotic behaviour of perturbative coefficients of the two-point function for $p^2\gg M^2$ is $\sim\sum_p p!\; g^p$, which accordingly gives a less severe ambiguity of $\mathcal{O}(\Lambda^2/p^2)$, where again one can note the similarity with the QCD off-shell $p^2\gg M^2$ quark correlation function case [5, 7].

It is in fact possible to go a step further and to characterize at arbitrary next orders the renormalon properties of the naive perturbative expansion of the $1/N$ GN mass gap.

[4] In ref. [1] appeared already the (earliest) discussion on *ultraviolet* renormalons in a field theory framework, which were shown to be harmless (Borel summable) for the asymptotically free GN model.

Consider the second order (exact) RG dependence of the effective coupling [11]:

$$g/g(q^2) \equiv f = 1 + b_0 g \ln \frac{q^2}{\mu^2} + \frac{b_1}{b_0} g \ln \left[f \frac{(1 + b_1/b_0 g f^{-1})}{1 + b_1/b_0 g} \right] \tag{2.11}$$

defining f recursively, with $g \equiv g(\mu)$ and the beta function $\beta(g) = -2b_0 g^2 - 2b_1 g^3 - \cdots$ (where for clarity we reintroduce the original coupling and RG coefficients, i.e. before rescaling of g). We can put eq. (2.5), with (2.11), directly into the form of a Borel integral, after a convenient change of variable [12], defining the (Borel) variable t as

$$b_0\, t = \frac{1 - f}{1 + b_1/b_0 g}\,. \tag{2.12}$$

Taking expression (2.6), but using now eq. (2.11), we find after some algebra

$$\begin{aligned} r_1^{RG2} = -\frac{\mu}{2M} \int_0^\infty dt e^{-\frac{\alpha t}{2}} (1 - b_0 t)^{-1-C} \Bigg[1 - \frac{\mu}{2M} e^{-\frac{\alpha t}{2}} (1 - b_0 t)^{-C} + \\ + \frac{\sqrt{\pi}}{2} \sum_{p=1}^{\infty} \frac{(\mu^2/4M^2)^p}{p!\Gamma[3/2 - p]}\, e^{-p\alpha t} (1 - b_0 t)^{-2pC} \Bigg] \end{aligned} \tag{2.13}$$

where $C = b_1/(2b_0^2)$ and $\alpha = 1/g + b_1/b_0$. To obtain (2.13) we expanded eq. (2.7) in powers of q^2/M^2, and used eqs. (2.11), (2.12). This gives the complete (leading and all subleading orders) series of infrared renormalons (initially corresponding to cuts at $b_0 t_p = 1/2, 1, ..(2p+1)/2$, $p \in \mathbf{N}^*$). The change of variable (2.12) makes calculations more convenient since expression (2.13) has only a cut at $t \geq 1/b_0$. Now we can calculate the ambiguity to all (perturbative) orders (of course still limited to $1/N$ order), which we define by the difference of contour above and below the cut. We find, again after some algebra:

$$\begin{aligned} (2\pi i)^{-1} \delta M^P = \pm \frac{\bar{\Lambda}}{2b_0} \Bigg[\frac{(2e)^{-C}}{\Gamma[1 + C]} - \frac{\bar{\Lambda}}{2M} \frac{e^{-2C}}{\Gamma[1 + 2C]} + \\ + \frac{\sqrt{\pi}}{2} (2e)^{-C} \sum_{p=1}^{\infty} \frac{(2p + 1)^{(2p+1)C}}{p!\Gamma[3/2 - p]\Gamma[1 + (2p + 1)C]} \left(\frac{(2e)^{-C} \bar{\Lambda}}{2M} \right)^{2p} \Bigg] \end{aligned} \tag{2.14}$$

where we used essentially

$$\lim_{\epsilon \to 0} [\int_0^{\infty + i\epsilon} - \int_0^{\infty - i\epsilon}] dt e^{-\alpha t} (1 - \beta t)^\gamma = 2\pi i\; e^{-\alpha/\beta} \beta^\gamma \alpha^{-(1+\gamma)} \Gamma[-\gamma]^{-1} \tag{2.15}$$

and identified the $\overline{MS}$ scale $\bar{\Lambda} = \mu e^{-1/(2b_0 g)} (b_0 g)^{-C} [1 + (b_1/b_0) g]^C$ consistently at second RG order. Eq. (2.14) thus gives the full series of ambiguities due to infrared renormalons (for the first graph in figure 1), in the form of power corrections of order Λ^p, with the first term in the bracket of (2.14) the leading order ambiguity of $\mathcal{O}(\Lambda)$. If $b_1 = 0$ (e.g. at first RG order) expression inside the bracket of eq. (2.10) simplifies to $(1 + r^2/4)^{1/2} - r/2$, $r \equiv \Lambda/M$ (indeed sufficient at this $1/N$ order, since from eq. (2.2) r_1 is already the $1/N$ term.).

As already emphasized, the above derivation of (2.14) only uses information that is in fact purely perturbative: the effective coupling at second RG order, eq. (2.11), and the specific GN mass q^2 kinematic dependence, eq. (2.7) of the (one-loop) skeletal first graph in figure 1. Accordingly, a similar derivation is possible for the QCD quark pole mass renormalon properties (indeed the equivalent of the information in (2.14) in the QCD case is also known, though perhaps expressed in a slightly different form [7]).

Actually, by considering only the first graph of figure 1 we slightly oversimplified the complete renormalon picture for the GN model: clearly, the second tadpole graph also gives renormalons, *if* considered purely perturbatively. These are easily analyzed similarly to eqs. (2.6)-(2.7), and the expanded integrand for $q^2 \to 0$ leads to renormalon poles at $b_0 t_p = p \in \mathbf{N}^*$. Incidentally, the pole at $b_0 t = 1$ exactly cancels the one in (2.13), but the leading as well as all subleading poles for $b_0 t_p \geq 3/2$ in (2.13) remain uncancelled.

In summary, we thus observe from the structure of (2.14) that even resumming the full integrand $1-\zeta$ (taking the full series of sub-leading renormalons) does not remove in any way the leading ambiguities (even if there are some "accidental" cancellations among subleading poles at this purely perturbative level, between the two graphs of figure 1 as above discussed). This is not at all surprising, since as emphasized (2.13), (2.14) are still perturbative calculations. This ambiguity in the purely perturbative piece of the pole mass is severe, being of the same $\mathcal{O}(\Lambda)$ order than the mass itself, similarly to the QCD pole mass [5] ambiguity. As explained above in detail, this originates from the $1-\zeta$ factor, specific to the pole mass, which has an expansion for $q^2 \to 0$ whose leading term is $\mathcal{O}(|q|^{-1})$, cf. eq. (2.7). It is interesting to compare those pole mass results with what happens for other related quantities of interest. For instance, the GN fermion condensate $\langle \bar\Psi \Psi \rangle$ (which is related to the mass gap, since e.g. $\langle \bar\Psi \Psi \rangle = N M_P$ in the $N \to \infty$ limit), has a leading ambiguity in its perturbative expansion which is much power suppressed. More precisely, the relevant part of the condensate can be obtained, roughly speaking, by closing the external leg of the first graph in figure 1. A straightforward calculation along the lines of eqs. (2.6)–(2.10) gives for the relevant integral:

$$\langle \bar\Psi \Psi \rangle \sim \int_0^{\mu^2} dq^2 \left[c_0 + c_1 \ln M + \mathcal{O}\left(\frac{q^2}{M^2}\right)\right] G\left[q^2\right] \tag{2.16}$$

where c_0 and c_1 are unessential constants not specified here. Accordingly, when takin in (2.16) only the perturbative part $g(q^2)$ of the propagator $G[q^2]$, cf. eq. (2.6), one finds for the Borel transform an IR renormalon pole at $b_0 t = 1$, thus associated with an ambiguity of $\mathcal{O}(\Lambda/\mu)^2$. This result for the IR renormalon in the perturbative tail of the GN condensate was in fact obtained long ago [13], both in this heuristic form and rigorously by using a lattice regularization with Wilson fermions, where it was precisely shown that this absence of a leading order renormalon is due to the chiral symmetry. As concluded by these authors, even if the perturbative expansion of the condensate is not, strictly speaking, Borel summable, the ambiguity may be neglected since it is power suppressed with respect to the NP condensate itself, the latter being of $\mathcal{O}(\Lambda)$. A similar suppression due to chiral symmetry of the leading IR renormalon also occurs for the QCD quark condensate in four dimensions.

JHEP01(2003)014

Coming back to the GN pole mass case, we have however to stress once more at this point that all the perturbative renormalon results derived in this section are in a sense artifacts, since the perturbative expansion that we considered was on purpose artificially generated as an incomplete part of the full $1/N$ expression of the mass gap.

3. Borel summability of the exact $\mathcal{O}(1/N)$ mass

Alternatively, since expression (2.5) is exact at $1/N$ order, we can calculate exactly the expression of the mass gap, i.e. without truncating (2.5) to its perturbative expansion. Indeed integral (2.5) can be evaluated analytically exactly: after a convenient change of variable $\zeta^{-1} = \tanh(\phi/2)$ we obtain

$$r_1 = \frac{1}{2}\left[Ei[-\theta] - \ln\theta - \gamma_E + \ln\left(\ln\frac{\mu^2}{M^2}\right) - 2\ln(\cosh[\theta/2]) + \ln\frac{\mu^2}{M^2}\right] \tag{3.1}$$

with $\chi = (1+4M^2/\mu^2)^{1/2} \equiv 1/\tanh(\theta/2)$ (i.e. $\theta = \ln[(\chi+1)/(\chi-1)] \ \geq 0$), and $Ei(-x) \equiv -\int_x^\infty dte^{-t}/t$ $(x \geq 0)$ the Exponential Integral function. The term $-2\ln(\cosh[\theta/2])$ in (3.1) corresponds to the (unsubtracted) tadpole graph of figure 1, and the terms $\ln\ln(\mu^2/M^2) \equiv -\ln g$ and $\ln\mu^2/M^2 \equiv 1/g$ are the subtraction terms for the first and tadpole graphs, respectively. One can easily check the finiteness of (3.1), if letting the "cutoff" $\mu \to \infty$. Now, as already stressed, here we are interested in the complete asymptotic behaviour, thus letting $\mu \gg \Lambda$ but kept finite, retaining eventually all the power correction terms in Λ/μ. The function $Ei(-x)$ for $x > 0$ has an asymptotic expansion with factorial but *sign alternating* coefficients, therefore explicitly Borel summable and perturbatively unambiguous. More precisely when re-expanding (3.1) in perturbation, we obtain

$$M^P = M\left[1 + \frac{1}{2N}\left(2\ln 2 - \gamma_E - \frac{M^2}{\mu^2}\left[\sum_{n=0}^{\infty}(-1)^n n!\, g^{n+1} + \mathcal{O}\left(\frac{M^2}{\mu^2}\right)\right]\right)\right] \tag{3.2}$$

where the higher order power correction terms, that we do not give explicitly here, can be obtained by a systematic expansion in M^2/μ^2 of θ. The tadpole graph in figure 1 contributes a constant term $2\ln 2$, relevant to the mass gap determination, but does not contribute the factorial asymptotic behaviour of the perturbative series, as already mentioned.

We emphasize that the specific $1-\zeta$ form of the integrand in (2.5) plays an essential role for the Borel summability of expression (3.2), which is accordingly peculiar to the pole mass. In contrast, the off-shell two-point function expression (2.3) for arbitrary p^2 may be evaluated similarly non-perturbatively (at $1/N$ order), and does not lead to a directly Borel summable perturbative series, in consistency with the results obtained and discussed previously in ref. [10].[5] We shall come back to this specificity of the pole mass in more details in next section 4.

We also stress that the use of the $\mathcal{O}(1/N)$ mass gap relation eq. (2.2) in the lagrangian mass $m \to 0$ limit, though it greatly simplifies the analytic evaluation of eq. (2.5), plays no particular role in the good asymptotic properties of eq. (3.2): more precisely, starting

[5] Technically, the simplest such terms can be expressed in terms of $Ei(\theta)$, which accordingly (since $\theta > 0$) has asymptotic expansion with same sign factorial coefficients and imaginary part $\pm i\pi$.

from the exact $1/N$ mass expression (2.5), and introducing the arbitrary mass dependence m, Borel summability is maintained with an asymptotic expansion similar to eq. (3.2) for any value of the pole mass $M_P \gg \Lambda$ (see appendix A).

Finally to obtain the correct mass gap $M^P/\bar{\Lambda}$ at next-to-leading $1/N$ order from expression (3.1), one should yet introduce the $\overline{MS}$ basic scale above defined after eq. (2.15). Dropping terms of higher $\mathcal{O}(1/N^2)$ order, we obtain

$$\frac{M^P}{\overline{\Lambda}} = 1 + \frac{r_1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) = 1 + \frac{1 + 2\ln 2 - \gamma_E}{2N} \tag{3.3}$$

in agreement with ref. [6]. Note however that our expression (3.1) differs in fact from ref. [6], more precisely by the $Ei[-\theta]$ term. This is simply because in ref. [6] all terms vanishing as inverse powers of μ were dropped, which is sufficient to identify the mass gap eq. (3.3). From our result, this is consistent because at the (non-perturbative) level of eq. (3.1), all those "power correction" contributions from $Ei(-\theta)$ can be unambiguously separated, thus dropped from eq. (3.2), to let only the part relevant to determine the mass gap. The explicit Borel summability of the "educated" perturbative expansion of the pole mass at $1/N$ order, eq. (3.2), confirms the consistency of the whole procedure. An equivalent signature of Borel summability here is the absence of positive powers of μ in (3.2), in contrast with the purely perturbative contribution eq. (2.8). Applying the same procedure to other quantities than the pole mass, one eventually ends up with asymptotic expansions with non sign-alternated factorial coefficients and μ^P power terms with $P > 0$ (as is clear from the above expression of θ, cf. footnote 5). Within a complete NP calculation, the latter μ^P terms in fact cancel when combining the OPE and perturbative parts, as illustrated in explicit calculations for some vacuum expectation value [8, 14] and off-shell correlation functions [9] of the $O(N)$ sigma model at $1/N$ order. As mentioned in introduction, this also illustrates that, though any complete NP calculation is expected to be renormalon ambiguity free, one cannot expect in general (for an arbitrary Green function) to be able to single out unambiguously the perturbative contributions.

Coming back to our result eq. (3.2), it appears thus immediately in apparent contradiction with those obtained starting from purely pertubative expansions, (2.8)–(2.14) above. We shall examine in next section how to reconcile these two different pictures.

4. Explicit disappearance of IR renormalons

How exactly the "bad" factorial coefficients with no sign alternation in eq. (2.8) disappear, or more precisely transmute into "good" sign-alternated factorials, in the exact expression (3.2)? Clearly, it can only be that this necessary change of the wrong sign perturbative expansion factorial coefficients should occur with the NP power expansion contributions: the weak point of the standard perturbative renormalon picture is that we have expanded the $1 - \zeta$ term in the infrared low q^2 regime, while keeping the short distance, perturbative effective coupling form of the propagator. This is motivated, as already mentioned, from the fact that the latter information is a priori the only accessible one in more involved theories such as QCD. In the present case, as we know exactly the mass at $1/N$ order one may at first hope to infer such transformations by examining e.g. the systematic short distance $q^2 \to \infty$ and long distance $q^2 \to 0$ power expansion of the integrand in (2.6), which are perfectly well-defined.[6]

In fact, to see the vanishing of renormalons operating needs a little more sophisticated analysis. Following e.g. refs. [8, 9], we introduce the Mellin-Barnes (MB) transform for a part of the integral eq. (2.5), which then takes the form (again omitting here the tadpole graph for simpler illustration):

$$r_1 = \frac{1}{4}\int_0^{\mu^2} \frac{dq^2}{M^2} \int_0^{\infty} dt \frac{1}{2\pi\, i} \oint ds K(s,t) \left(\frac{M^2}{q^2}\right)^{-s} \tag{4.1}$$

where the Kernel (inverse MB transform) is defined in our case by

$$K(s,t) = \int_0^{\infty} dx x^{s-1} (1-\zeta)\, \zeta^{-1} \left[\frac{\zeta-1}{\zeta+1}\right]^t . \tag{4.2}$$

The MB transform method main purpose is that it will exhibit precisely the singularities of the integrand, in the Borel plane t of interest. The sequel is just algebraic manipulation. Changing again variable $\zeta^{-1} = \tanh(\frac{\phi}{2})$, *except* for the $1-\zeta$ term kept on purpose, using eq. (2.7), as an expansion in q/M, (4.2) can be evaluated exactly, and

$$\int_0^{\mu^2} \frac{dq^2}{4M^2} \left(\frac{M^2}{q^2}\right)^{-s} K(s,t) = \frac{\mu^2}{2M^2} \sum_{a\geq -1} 2^{-a} c_a \frac{\Gamma[1+a-2s]\Gamma[-a/2+s+t]}{\Gamma[1+a/2-s+t](1+s)} \left(\frac{M^2}{\mu^2}\right)^{-s} \tag{4.3}$$

defined for $\mathrm{Re}[s+t] > a/2$ and $\mathrm{Re}[2s] < 1+a$. The latter conditions are such that integral (4.2) converges, and play essential role in determining the singularities. The variable a in (4.3) is simply the power of $q/M \sim \sinh\phi$ in expansion (2.7), with coefficient c_a respectively. Thus $a = -1$ with $c_{-1} = -2$ corresponds to the leading renormalon, and $a = 0, ..2p+1$, $p \in \mathbf{N}$ to subleading ones. To evaluate the s integral one can simply close the contour on the left, and sum over residues of the poles included in this domain (since $x^{-s} \equiv (M^2/q^2)^{-s}$ decays exponentially fast for the asymptotic regime $q^2 \gg M^2$ we are interested in). Expression (4.3) has (simple) poles at $s = -1$, $s = a/2 - t - k$, and $2s = a+1+k$, $k \in \mathbf{N}$, where the latter poles do not contribute for the relevant contour. The final result is

$$r_1 \sim \frac{\mu^2}{4M^2} \int_0^{\infty} dt \sum_{a\geq -1} \frac{c_a}{2^a} \left[\frac{\Gamma[3+a]\Gamma[t-1-a/2]}{\Gamma[2+t+a/2]} \frac{M^2}{\mu^2} + \right. \tag{4.4}$$
$$\left. + e^{-t/g} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[1+2t+2k]}{k!\Gamma[1+2t+k](1+a/2-t-k)} \left(\frac{M^2}{\mu^2}\right)^{k-a/2} \right]$$

where the first term in bracket corresponds to the residue of the pole at $s = -1$, while other terms correspond to the sum over residues of the poles at $s = a/2 - t - k$, and we used $(M^2/\mu^2)^t \equiv e^{-t/g}$. In (4.4) one sees that the first term in the bracket and the

[6] e.g. the infrared $q^2 \to 0$ expansion of the dressed propagator is purely polynomial, with no log. dependence: $[\zeta \ln[\frac{\zeta+1}{\zeta-1}]]^{-1} \simeq \frac{1}{2} - \frac{q^2}{24M^2} + \cdots$.

summed terms both have poles at $t(a,k) = 1 + a/2 - k$, which can occur at $t > 0$ depending on a, k values. On more physical grounds, the contributions from the first term, the initially $s = -1$ pole, originate from power terms $(M^2/q^2)^{-s}$ in eq. (4.1), and correspond intuitively to non-perturbative "OPE" contributions, while the perturbative contributions are those multiplied by $e^{-t/g}$ in (4.4). Even though in fact both contributions are in the end of the same (non-perturbative) order, such as to compensate the bad poles in $t > 0$, it is natural within the Borel transform formalism to distinguish those two contributions as "non-perturbative" and "perturbative" respectively, since typically the second contribution has the characteristic t-dependent overall factor $e^{-t/g} = (M^2/\mu^2)^t$ multiplying an expression with poles at $t > 0$, so that when this latter contribution is considered alone it organizes as an ordinary perturbative expansion when integrated over t. More precisely, for example the poles at $t = 1/2$ for $a = -1$ correspond to the leading order renormalon, with $k = 0$. Indeed, keeping only the leading renormalon perturbative terms, $\propto e^{-t/g}$, for $k = 0$, one recovers exactly eq. (2.9).

Now it can be easily checked that this $t = 1/2$ pole in fact cancels exactly against the first NP term $t = 1/2$ pole, and similarly for all subleading poles at $t = 1, 3/2, ..(2p+1)/2$. This is the announced disappearance of renormalons. Moreover all cancellations happen, for a given pole at $t = 1+a/2-k$, between NP and perturbative terms of the same k values. Since (4.4) is in the form of a Borel integral, and after cancellations all remaining poles occur at $t < 0$, it defines a Borel summable series, whose leading terms just correspond to the asymptotic series defined in (3.2). To see that, it is simpler to alternatively proceed directly with (4.2) using the change of variable $\zeta \to \phi$ for any terms in the integrand, i.e. without going through power expansion (2.7): after MB transform one ends up directly with a Borel integral where no poles at all appear at $t > 0$, and which exactly gives the asymptotic series in eq. (3.2), including correct finite terms $-\gamma_E + \cdot$. (N.B. there are also poles at $t = 0$ in (4.4), which as usual [7] simply corresponds to the UV divergences, and are removed consistently by the appropriate subtraction terms, that we do not display explicitly.) For instance, illustrating only the terms from the first graph in figure 1, after cancellations of the $t > 0$ poles the MB transformation gives

$$
\begin{aligned}
r_1 &\sim -\frac{\mu^2}{2M^2}\int_0^\infty dt \left[\frac{M^2}{\mu^2}\frac{1}{t(1+t)} + e^{-t/g}\left[-\frac{1}{t}\frac{M^2}{\mu^2} + \frac{3+2t}{1+t}\left(\frac{M^2}{\mu^2}\right)^2 + \mathcal{O}\left(\frac{M^2}{\mu^2}\right)^3\right]\right] \\
&\sim -\frac{1}{2}\left[\gamma_E + \frac{M^2}{\mu^2} g \sum_{n=0}^{\infty}(-1)^n n! g^n\right] \qquad (4.5)
\end{aligned}
$$

to compare with eq. (3.2) (where we used e.g. $\int_0^\infty dt\, t^{-1}[1/(t+1) - e^{-t}] \equiv \gamma_E$). Note that in eq. (4.5) all the "bad" positive powers of μ/M, that signalled the non-Borel summability in the corresponding incomplete purely perturbative expressions eqs. (2.9), (2.13), have now disappear. Of course, proceeding in this "direct" way is nothing but a consistency check that the MB transform gives a correct alternative calculation of the asymptotic expansion of the exact integral eq. (2.5), which we started from anyway. But our explicit separation of the expanded perturbative renormalon part (2.7) in connection with the MB method allows to visualize explicitly the renormalon disappearance order by order in (4.4). For completeness note that a very similar MB transform analysis can be performed for the tadpole graph renormalons, which disappear similarly from a detailed analysis that we do not display here.

It may be of interest to examine the influence of the regularization scheme on the results derived above. As expected, the renormalon properties are independent of the regularization used, but they manifest themselves in a quite different form e.g. in dimensional regularization [8, 7]. The momentum cutoff regularization prescription considered here, introducing an explicit factorization scale μ, cf. eq. (2.5), is clearly very convenient for our purpose since it directly displays the NP power correction terms, while the interpretation of the latter for instance in dimensional regularization (DR) is not completelly obvious. Nevertheless, the appearance (and disappearance) of IR renormalons as discussed here can also be seen equivalently in DR. As is known from general analysis [8, 7], the IR renormalon ambiguities in DR manifest themselves in the fact that the Borel integral over the perturbative contributions has to be evaluated for complex ϵ, and gives a different result depending on whether $\mathrm{Im}[\epsilon] > 0$ or $\mathrm{Im}[\epsilon] < 0$. The difference between the two results is proportional to the NP contribution. In a rather simplified analysis of the GN pole mass (see appendix B for more detail), considering the equivalent of the MB plus Borel transform eqs. (4.1)–(4.4) in DR, one arrives at an expression quite similar to eq. (4.4), but in which now both terms have poles at $t = a/2 + 1 - k - \epsilon$, and their residue cancel each others order by order in a way completely equivalent to what was discussed above for $\epsilon \to 0$.

To conclude this section, going back to the more convenient cutoff regularization, one may alternatively understand perhaps more qualitatively the specific pole mass renormalon disappearance by examining the asymptotic behaviour of the off-shell mass expression (2.3), first expanded in powers of M^2/p^2, and proceeding with the MB transform similarly to the above described procedure. (This is then a completelly similar calculation than the one performed in details in ref. [9] for the $O(N)$ sigma model.) Skipping many details, we simply indicate sketchily that for the GN model, it would give coefficients of the $(M^2/p^2)^n$ terms with a form similar to eq. (4.3), but with essentially the replacements $\mu^2 \to p^2$; $(1+s)^{-1} \to \sum_i (1+s+i)^{-1}$, $0 \leq i \leq n-1$. Those poles at $s = -1-i$, of residues $\sim \Gamma[t-1-i]$, produce in turn poles at (integer) $t = 1, 2, \ldots, n$. Again, the integral over t may be cast into a Borel transform formally similar to (4.4), identifying a NP part and a perturbative part. However, in this off-shell case, the specific poles at $t_0 = 1, 2, \ldots$ within perturbative terms of order $(M^2/p^2)^n$, are cancelled by NP term of different order $n + t_0$. Since individual terms are singular, one cannot truncate the power expansion unless a definite integration contour prescription to avoid the poles is defined [9]. In other words, the separation between the NP and perturbative part in this $p^2 \neq M^2$ case is ambiguous, which means the non Borel summability of the perturbative series of the general two-point function (2.3) for arbitrary p^2, as obtained from a direct calculation, cf. remarks in section 3 (footnote 5). In contrast, for the pole mass any $(M^2/p^2)^n$ terms are replaced by 1, which "flatten" all orders of the M^2/p^2 expansion, so that the different cancellations of $t > 0$ poles now occur all at once.

5. Conclusion and perspectives for QCD?

In this paper we have exhibited in details the non-trivial disappearance of the "naive" perturbative IR renormalons of the GN mass gap at order $1/N$, implying the direct Borel summability of the "educated" perturbative expansion, as defined by eq. (3.2). Given the detailed renormalon vanishing mechanism, we are also confident that it should work similarly beyond $1/N$ order, though an explicit check has not been attempted. Concerning strictly our $1/N$ results for the GN pole mass, a number of additional remarks may be useful.

First, we should perhaps again emphasize that the result eq. (3.2) may be not a surprising one, as it could have been easily extracted from previous analysis of e.g. refs. [10, 6], if not explicitly displayed there. As motivated in introduction, our main purpose here was to illustrate in a calculable model how the NP contributions to the pole mass organize to eliminate completely the renormalon artifacts, even though the latter are somewhat artificially introduced by defining the naive perturbation theory, similar to the perturbative expansion of QCD quark pole masses.

Second, the (would-be) perturbative ambiguities due to IR renormalons for the pole mass here discussed are severe, being of the same ($\mathcal{O}(\Lambda)$ order than the pole mass itself. In contrast, there are other related quantities of interest, like most typically the GN model fermion condensate, where the leading order potentially dangerous renormalon singularities in the perturbative part disappear [13] by explicit cancellations due to the chiral symmetry, as reexamined briefly in the end of section 3. There, the cancellation of renormalons is such that the actual perturbative ambiguities are of higher order than the condensate dimension, thus power suppressed and to be considered harmless. Though the latter results are not unrelated with the present study, note that the IR renormalon cancellation mechanism as we exhibited here is of a rather different nature, since *all* (but not only the leading) IR renormalon singularities are annihilated each other between the purely perturbative and NP contributions, see e.g. eq. (4.4). But we stress again that these results are very specific to the pole mass.

It is thus tempting to speculate briefly on the possibility of ultimate (NP) renormalon disappearance in QCD similarly (to some extent) to the mechanism discussed here. QCD in the massless quark limit also has a mass gap, since the approximate chiral symmetry of the light quark sector is dynamically broken. Now *if* we assume as usual [7] that the dominant IR renormalon contributions to the pole mass is given by graphs of a topology analogous to the first one in Fig 1, but with a gluon propagator (wavy line) dressed now with massive, constituent quarks of mass $M_Q \sim \Lambda_{QCD} = \bar{\Lambda}$, one may expect this propagator to behave in the infrared in a way similar to eq. (2.4). (Assuming also that such quark loops are complemented with appropriate QCD gauge sector contributions, so to match the correct beta function in a gauge invariant way, as indeed usually assumed in most QCD infrared renormalon issues [7]). We see no reason why such assumption on the NP behaviour would not be consistent with the usual perturbative behaviour, and in particular with the standard heavy quark pole mass renormalon picture [5], for which $M_Q \gg \bar{\Lambda}$. This is also irrespective of the fact that confinement in QCD ultimately makes the pole quark mass relevance somewhat elusive. What can be still theoretically relevant would be to have in this way a procedure to evaluate the NP contributions to the light constituent quark masses $M_Q \sim \bar{\Lambda}$ from first principles (and perhaps more interestingly for the NP order parameters related to chiral symmetry breaking). A detailed QCD analysis is however beyond the present scope and left for future work.

Let us finally mention that, indeed, there exist more "direct" ways of modifying the asymptotic properties of the perturbative expansion of e.g. the mass gap, generically in asymptotically free models. It relies only on the purely perturbative information, but is based on a modification of the usual perturbative series. Such a method [15] appears to bypass the explicit disappearance of renormalons here exhibited, by directly removing the perturbative renormalon divergences, at least for adequate ranges of an (alternative) perturbative expansion parameter values compatible with the chiral limit.

A. Asymptotic behaviour for $M_P \gg \Lambda$

As mentioned in section 3 the use of the mass gap relation eq. (2.2), strictly valid only at $\mathcal{O}(1/N)$ and in the chiral limit (lagrangian mass $m \to 0$), plays in fact no particular role in the asymptotic properties eq. (3.2), of the exact $1/N$ pole mass. Consider eq. (2.5), but now for $M_P \gg \Lambda$ (which illustrates e.g. the massive theory case with $M_P \sim m \gg \Lambda$): it takes the form

$$r_1 = \int d\phi \left[e^{-\frac{\phi}{2}} \cosh \frac{\phi}{2} - \frac{\phi}{2} \left(1 + \ln \left(\frac{M_P}{\Lambda} \right) \right)^{-1} \right] \left(\ln \left(\frac{M_P^2}{\Lambda^2} \right) + \frac{\phi}{\tanh \frac{\phi}{2}} \right)^{-1} \tag{A.1}$$

to be expanded in powers of $\ln^{-1}(M_P^2/\Lambda^2)$, where we used again $\phi = \ln[(\zeta+1)/(\zeta-1)]$. Each coefficient of such an expansion contains a logarithmic divergence, renormalizable by subtraction (removing essentially the divergent $(1+\phi)/2$ piece in the bracket of eq. (A.1)). After some algebra we find finally that the renormalized series has the leading asymptotic behaviour

$$r_1^{(as)}(M_P \gg \Lambda) \sim \sum_{p=0}^{\infty} (-1)^{p+1} \frac{p!}{\ln^{p+1}(M_P^2/\Lambda^2)} \tag{A.2}$$

which qualitatively agrees asymptotically with eq. (3.2), provided one identifies $M_P \sim \mu \gg \Lambda$, i.e. $\ln(M_P^2/\Lambda^2) \sim 1/g$. The sign alternation in (A.2), leading to Borel summability, again makes the main difference with the behaviour obtained starting from the naive perturbative analysis of section 3.

B. IR renormalons (dis)appearance in dimensional regularization

In dimensional regularization (DR) with $D = 2 - \epsilon$ (so that the GN model is renormalizable for $\mathrm{Re}[\epsilon] > 0$), eq. (2.5) may be cast in the form (omitting once more the tadpole graph to simplify):

$$r_1 = \frac{gN}{8\pi} \left(\frac{\mu}{M} \right)^{\epsilon} \int \frac{dq^2}{M^2} \left(\frac{q^2}{M^2} \right)^{-\epsilon/2} (1 - \zeta)\, G\left[q^2, M^2, \epsilon \right] \tag{B.1}$$

where $\zeta \equiv (1+4M^2/q^2)^{(1/2)}$, and the DR regularized auxiliary field propagator

$$G[q^2M^2,\epsilon] = \left[1 - \frac{g_0 N}{2\pi}(1-\epsilon)\Gamma[\epsilon/2]\int_0^1 dx\,\left(M^2+q^2x(1-x)\right)^{(-\epsilon/2)}\right]^{-1} + \mathcal{O}\left(\frac{1}{N}\right) \quad \text{(B.2)}$$

has the $\epsilon \to 0$ limit eq. (2.4) (of course after coupling renormalization $g_0 = Z_g g$, which details are not explicitly displayed, absorbing the divergence in $\Gamma[\epsilon/2]$).

Skipping the detailed calculation and unessential overall factors, the important point is that the large q^2 behaviour of eq. (B.2) in DR, equivalent of eq. (2.6), replaces integrals over $\ln q^2$ with integral of the form[7]

$$\sum_{a\geq -1}\sum_{l}\int_0^{\mu^2}\frac{dq^2}{M^2}\left(\frac{q^2}{M^2}\right)^{a/2-\epsilon/2-\epsilon\, l/2} \quad \text{(B.3)}$$

(where the sum over a comes from the expansion of $1-\zeta$, similarly to eq. (2.7), while the sum over l simply comes from the geometic expansion of eq. (B.2)). Though integration of (B.3) for finite ϵ does not lead to factorial divergences unlike its $\epsilon \to 0$ equivalent eq. (2.6), it gives poles at $1+a/2-\epsilon(1+l)/2$ to be evaluated by taking ϵ complex, and the $\epsilon \to 0$ result differs for $\mathrm{Im}[\epsilon] > 0$ or $\mathrm{Im}[\epsilon] < 0$. This is similar to general results [8, 7] on the way in which renormalon ambiguities manifest themselves in DR. Going now to Mellin and Borel transform as examined for the cutoff regularization in section 4, the DR equivalent of eq. (4.4) takes the form [8]

$$r_1^{DR} \sim \int_0^\infty dt \sum_{a\geq -1}\left[\frac{\Gamma[3+a-\epsilon]\Gamma[t-\epsilon/2-1-a/2]}{\Gamma[2+t+a/2-\epsilon/2]}\right. \quad \text{(B.4)}$$
$$\left. + e^{-t/g}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[1+2t+2k]}{k!\Gamma[1+2t+k](1+a/2-t-k-\epsilon/2)}\left(\frac{M^2}{\mu^2}\right)^{k-a/2-1+\epsilon/2}\right]$$

where the first and second term in bracket occur as residues of poles at $s=-1+\epsilon/2$ and $s = a/2-t-k$, respectively, where s is the Mellin transform variable, cf. eq. (4.1). Both terms have poles at $t = a/2+1-k-\epsilon/2$, and their residue cancel each others order by order in a way completely similar to eq. (4.4) for $\epsilon \to 0$.

References

[1] D.J. Gross and A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, *Phys. Rev.* **D 10** (1974) 3235.

[2] R.F. Dashen, B. Hasslacher and A. Neveu, *Semiclassical bound states in an asymptotically free theory*, *Phys. Rev.* **D 12** (1975) 2443.

[7]Note that, even if DR regularizes the momenta integrals, we define the integral over the IR region $0 < q^2 < \mu^2$ since it is the relevant one for IR renormalons.

[8]To simplify without much loose of generality we can in fact neglect for our purpose in the Mellin-Borel transform the detailed $\epsilon\, l/2$ dependence of the auxiliary field propagator in eqs. (B.2), (B.3).

[3] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Ann. Phys. (NY)* **120** (1979) 253; *Relativistic factorized S matrix in two-dimensions having O(N) isotopic symmetry*, *Nucl. Phys.* **B 133** (1978) 525.

[4] P. Forgacs, F. Niedermayer and P. Weisz, *The exact mass gap of the Gross-Neveu model, 1. The thermodynamic Bethe ansatz*, *Nucl. Phys.* **B 367** (1991) 123.

[5] M. Beneke and V.M. Braun, *Heavy quark effective theory beyond perturbation theory: renormalons, the pole mass and the residual mass term*, *Nucl. Phys.* **B 426** (1994) 301 [`hep-ph/9402364`];
I.I.Y. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, *The pole mass of the heavy quark. perturbation theory and beyond*, *Phys. Rev.* **D 50** (1994) 2234 [`hep-ph/9402360`].

[6] P. Forgacs, F. Niedermayer and P. Weisz, *The exact mass gap of the Gross-Neveu model, 2. The $1/N$ expansion*, *Nucl. Phys.* **B 367** (1991) 157.

[7] For a review, see M. Beneke, *Renormalons*, *Phys. Rept.* **317** (1999) 1 [`hep-ph/9807443`].

[8] F. David, *Nonperturbative effects and infrared renormalons within the $1/N$ expansion of the O(N) nonlinear sigma model*, *Nucl. Phys.* **B 209** (1982) 433: *On the ambiguity of composite operators, i.r. renormalons and the status of the operator product expansion*, *Nucl. Phys.* **B 234** (1984) 237.

[9] M. Beneke, V.M. Braun and N. Kivel, *The operator product expansion, non-perturbative couplings and the Landau pole: lessons from the O(N) sigma-model*, *Phys. Lett.* **B 443** (1998) 308 [`hep-ph/9809287`].

[10] M. Campostrini and P. Rossi, *Dimensional regularization in the $1/N$ expansion*, *Int. J. Mod. Phys.* **A 7** (1992) 3265

[11] C. Arvanitis, F. Geniet, M. Iacomi, J.L. Kneur and A. Neveu, *Variational solution of the Gross-Neveu model: finite N and renormalization*, *Int. J. Mod. Phys.* **A 12** (1997) 3307 [`hep-th/9511101`];
J.-L. Kneur, *Variational quark mass expansion and the order parameters of chiral symmetry breaking*, *Phys. Rev.* **D 57** (1998) 2785 [`hep-ph/9609265`].

[12] G. Grunberg, *Renormalons and fixed points*, *Phys. Lett.* **B 372** (1996) 121 [`hep-ph/9512203`].

[13] F. David and H.W. Hamber, *Chiral condensate with Wilson fermions*, *Nucl. Phys.* **B 248** (1984) 381.

[14] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Two-dimensional sigma models: modeling nonperturbative effects of quantum chromodynamics*, *Phys. Rept.* **116** (1984) 103.

[15] J.L. Kneur and D. Reynaud, *Borel convergence of the variationally improved mass expansion and dynamical symmetry breaking*, *Eur. Phys. J.* **C 24** (2002) 323 [`hep-th/0107073`]; *(Borel) convergence of the variationally improved mass expansion and the O(N) Gross-Neveu model mass gap*, *Phys. Rev.* **D 66** (2002) 085020 [`hep-th/0205133`].